

# TENSOR PRODUCT SURFACES AND LINEAR SYZYGIES

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**ABSTRACT.** Let  $U \subseteq H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b))$  be a basepoint free four-dimensional vector space, with  $a, b \geq 2$ . The sections corresponding to  $U$  determine a regular map  $\phi_U : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ . We show that there can be at most one linear syzygy on the associated bigraded ideal  $I_U \subseteq k[s, t; u, v]$ . Existence of a linear syzygy, coupled with the assumption that  $U$  is basepoint free, implies the existence of an additional “special pair” of minimal first syzygies. Using results of Botbol [1], we show that these three syzygies are sufficient to determine the implicit equation of  $\phi_U(\mathbb{P}^1 \times \mathbb{P}^1)$ .

## 1. INTRODUCTION

A tensor product surface is the image of a map  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ . Such surfaces arise in geometric modeling, and it is often useful to find the implicit equation for the surface. Standard tools such as Gröbner bases and resultants tend to be slow, and the best current methods rely on Rees algebra techniques. The use of such methods was pioneered by the geometric modeling community (e.g. Sederberg-Chen [16], Sederberg-Goldman-Du [17], Sederberg-Saito-Qi-Klimaszewski [18], Cox-Goldman-Zhang [9]). Further work on using Rees algebras in implicitization appears in Busé-Jouanolou [3], Busé-Chardin [4], Botbol [1] and Botbol-Dickenstein-Dohm [2]; see Cox [7] for a nice overview. A key tool is the approximation complex  $\mathcal{Z}$ , introduced by Herzog-Simis-Vasconcelos in [13], [14].

Let  $R = k[s, t, u, v]$  be a bigraded ring over an algebraically closed field  $k$ , with  $s, t$  of degree  $(1, 0)$  and  $u, v$  of degree  $(0, 1)$ . Let  $R_{a,b}$  denote the graded piece in bidegree  $(a, b)$ . A regular map  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$  is defined by four polynomials

$$U = \text{Span}\{p_0, p_1, p_2, p_3\} \subseteq R_{a,b} = H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b))$$

with no common zeros on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $I_U = \langle p_0, p_1, p_2, p_3 \rangle \subset R$ ,  $\phi_U$  be the associated map  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$  and  $X_U = \phi_U(\mathbb{P}^1 \times \mathbb{P}^1) \subseteq \mathbb{P}^3$ . The assumption that  $U$  is basepoint free means that  $\sqrt{I_U} = \langle s, t \rangle \cap \langle u, v \rangle$ . Motivated by [8], in [15], Schenck-Seceleanu-Validashti show that for tensor product surfaces of bidegree  $(2, 1)$ , the existence of a linear syzygy on  $I_U$  imposes very strong conditions on  $X_U$ . We show this is not specific to the bidegree  $(2, 1)$  case. Our main result is:

**Theorem:** If  $a, b \geq 2$  and  $U$  is basepoint free, then there is at most one linear first syzygy on  $I_U$ . A linear first syzygy gives rise to a special pair of additional first syzygies. These three syzygies determine the degree  $(2a - 1, b - 1)$  component of the approximation complex  $\mathcal{Z}$ . By [1], the determinant of the resulting square matrix is a power of the implicit equation of  $X_U$ .

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**Example 1.1.** Suppose  $(a, b) = (2, 2)$ , and

$$U = \text{Span}\{t^2u^2 + s^2uv, t^2uv + s^2v^2, t^2v^2, s^2u^2\} \subseteq H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)),$$

which has a first syzygy of bidegree  $(0, 1)$ . A computation shows that  $I_U$  has seven minimal first syzygies, in bidegrees

$$(0, 1), (2, 1), (2, 1), (0, 3), (2, 2), (4, 1), (6, 0).$$

By Theorem 2.2, the three syzygies of bidegree  $(0, 1), (2, 1), (2, 1)$  are generated by the columns of

$$\begin{bmatrix} v & 0 & s^2u \\ -u & -t^2v & 0 \\ 0 & t^2u + s^2v & 0 \\ 0 & 0 & -t^2u - s^2v \end{bmatrix},$$

and the bidegree  $(2a - 1, b - 1) = (3, 1)$  component of the first differential in the approximation complex is

$$\begin{bmatrix} x_0 & 0 & 0 & 0 & x_2 & 0 & -x_3 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & x_2 & 0 & -x_3 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & x_0 \\ 0 & 0 & x_0 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & -x_1 & 0 & x_2 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & x_0 & 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 & x_2 & 0 & -x_3 \end{bmatrix}$$

The determinant of this matrix is

$$(x_0^3x_2 + x_1^3x_3 - x_0^2x_1^2)^2.$$

By Corollary 2.3 this means the implicit equation defining  $X_U$  is  $x_0^3x_2 + x_1^3x_3 - x_0^2x_1^2$ , and  $\phi_U$  is  $2 : 1$  by Lemma 1.4. By Corollary 2.4 the codimension one singular locus of  $X_U$  contains  $\mathbf{V}(x_0, x_1)$ ; in fact, in this case equality holds.

**1.1. Algebraic tools.** Two results from previous work will be especially useful; for additional background on approximation complexes and bigraded commutative algebra, see [15].

**Lemma 1.2.** [15] *If  $I_U$  has a linear first syzygy of bidegree  $(0, 1)$ , then*

$$I_U = \langle pu, pv, p_2, p_3 \rangle,$$

*where  $p$  is homogeneous of bidegree  $(a, b - 1)$ .*

A similar result holds if  $I_U$  has a first syzygy of degree  $(1, 0)$ . The lemmas below (Lemmas 7.3 and 7.4 of Botbol [1]) also play a key role. Botbol notes that the local cohomology module  $(H_2)_{4a-1, 3b-1}$  has dimension equal to the sum of the multiplicities at the basepoints, so if  $U$  is basepoint free, this module vanishes.

**Lemma 1.3.** [1] *Suppose  $a \leq b$ . If  $\nu = (2a - 1, b - 1)$ , then the determinant of the  $\nu$  strand of the approximation complex is of degree  $2ab - \dim(H_2)_{4a-1, 3b-1}$ .*

**Lemma 1.4.** [1] *If  $U$  has basepoints with multiplicities  $e_x$ , then*

$$\deg(\phi_U) \deg(F) = 2ab - \sum e_x, \text{ where } \langle F \rangle = I(X_U).$$

If  $U$  is basepoint free, the determinant of the  $\nu$  strand is the determinant of  $(d^1)_\nu$ .

## 2. PROOFS OF MAIN THEOREMS

**Theorem 2.1.** *If  $a, b \geq 2$  and  $U$  is basepoint free, then there can be at most one linear first syzygy on  $I_U$ .*

*Proof.* Suppose  $L$  is a linear syzygy of bidegree  $(0, 1)$  on  $I_U$ . By Lemma 1.2, we may assume

$$I_U = \langle pu, pv, p_2, p_3 \rangle = \langle p_0, p_1, p_2, p_3 \rangle,$$

where  $p$  is homogeneous of bidegree  $(a, b - 1)$ . Suppose there is another minimal first linear syzygy of bidegree  $(0, 1)$

$$\sum_{i=0}^3 p_i \cdot (a_i u + b_i v) = 0.$$

Let

$$\begin{aligned} \tilde{p}_2 &= \sum a_i p_i \\ \tilde{p}_3 &= \sum b_i p_i, \end{aligned}$$

so  $\tilde{p}_2 u + \tilde{p}_3 v = 0$ . But the syzygy module on  $[u, v]$  is generated by  $[v, -u]$ , so we must have  $\tilde{p}_2 = qv, \tilde{p}_3 = -qu$  for some  $q$  of bidegree  $(a, b - 1)$ . If in addition

$$\det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \neq 0, \text{ then}$$

$$I_U = \langle pu, pv, \tilde{p}_2, \tilde{p}_3 \rangle = \langle pu, pv, qu, qv \rangle.$$

Recall [12] that curves  $\mathbf{V}(f)$  of bidegree  $(a, b)$  and  $\mathbf{V}(g)$  of bidegree  $(c, d)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  sharing no common component meet in  $ad + bc$  points. If  $p$  and  $q$  share a common factor, then clearly  $I_U$  is not basepoint free; if they do not share a common factor, then  $\mathbf{V}(p, q)$  consists of  $2ab - 2a$  points; since  $a, b \geq 2$ , this again forces  $I_U$  to have basepoints. The same argument works if the additional syzygy is of bidegree  $(1, 0)$ , save that in this case since  $q$  is of degree  $(a - 1, b)$ ,  $\mathbf{V}(p, q)$  consists of  $2ab - a - b + 1$  points, and again  $I_U$  is not basepoint free.

Next, suppose

$$\det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} = 0.$$

If  $a_2 = a_3 = b_2 = b_3 = 0$ , then the second minimal first syzygy involves only  $pu$  and  $pv$ . If the syzygy is of bidegree  $(0, 1)$ , then by Lemma 1.2,  $(pu, pv) = (qv, qu)$ . Thus

$$pu = qv \implies p = fv, q = fu \implies fv^2 = fu^2,$$

a contradiction. If the syzygy is of bidegree  $(1, 0)$ , then  $(pu, pv) = (qs, qt)$ , and

$$pu = qs \implies p = fs, q = fu \implies fsv = fut,$$

again a contradiction.

Finally, if

$$\det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} = 0,$$

and  $a_2, a_3, b_2, b_3$  are not all zero, then  $c \cdot [a_2, b_2] = [a_3, b_3]$  for some  $c \neq 0$ , so letting  $\tilde{p}_2 = p_2 + cp_3$ , we may assume the syzygy involves only  $pu, pv, \tilde{p}_2$ . But then

$$pu(a_0 u + b_0 v) + pv(a_1 u + b_1 v) + \tilde{p}_2(a_2 u + b_2 v) = 0,$$

hence  $\tilde{p}_2$  is divisible by  $p$ . Arguing as above, this implies that  $I_U$  has basepoints. The same argument also works if the additional syzygy is of bidegree  $(1, 0)$ .  $\square$

**Theorem 2.2.** *If  $U$  is basepoint free,  $a, b \geq 2$  and there is a linear syzygy  $L$  of bidegree  $(0, 1)$  on  $I_U$ , then there are two additional first syzygies  $S_1, S_2$  of bidegree  $(a, b - 1)$ , such that*

$$\dim \text{Span}\{L, S_1, S_2\}_{(2a-1, b-1)} = 2ab.$$

*Proof.* By Lemma 1.2 we may assume  $(p_0, p_1) = (pu, pv)$ . Write  $p_2 = g_2v + f_2u$ . Then  $f_2p_0 + g_2p_1 - pp_2 = 0$ , so the kernel of  $[pu, pv, p_2]$  contains the columns of the matrix

$$M = \begin{bmatrix} v & f_2 \\ -u & g_2 \\ 0 & -p \end{bmatrix}.$$

In fact,  $M$  is the syzygy matrix of  $[pu, pv, p_2]$ : the sequence  $\{pu, p_2\}$  is not regular iff the two polynomials share a common factor. If  $u|p_2$ , then let  $p'_2 = p_2 + pv$ ;  $u|p'_2$  or  $p|p'_2$  imply  $I_U$  is not basepoint free. So the depth of the ideal of  $2 \times 2$  minors of  $M$  is two and exactness follows from the Buchsbaum-Eisenbud criterion [10]. Writing  $p_3 = f_3u + g_3v$ , the syzygy module of  $I_U$  contains the columns of  $N = \text{Span}\{L, S_1, S_2\}$ , where

$$N = \begin{bmatrix} v & f_2 & f_3 \\ -u & g_2 & g_3 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}.$$

As the bottom  $3 \times 3$  submatrix of  $N$  is upper triangular,  $\{L, S_1, S_2\}$  span a free  $R$ -module. The linear syzygy  $L$  is of bidegree  $(0, 1)$ , so in the degree  $\nu$  strand of the approximation complex it gives rise to

$$h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2a - 1, b - 2)) = 2a(b - 1)$$

columns of the matrix of the first differential  $d^1$ . The two syzygies  $S_1, S_2$  of bidegree  $(a, b - 1)$  each give rise to

$$h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a - 1, 0)) = a$$

columns of the matrix of  $d^1$ . That the columns are independent follows from the fact that  $\{L, S_1, S_2\}$  span a free  $R$ -module. Hence, these syzygies yield  $2ab$  columns the degree  $\nu$  component of the matrix of  $d^1$ .  $\square$

For Theorem 2.1 and Theorem 2.2 to hold, we need  $a, b \geq 2$ , even if  $U$  is basepoint free. If either  $a$  or  $b$  is at most one, there can additional minimal linear syzygies. For example, if  $(a, b) = (1, 1)$ , then there are four minimal linear first syzygies. However, it is easy to see that the theorems both hold if  $L$  is of bidegree  $(1, 0)$ .

**Corollary 2.3.** *If  $a, b \geq 2$ ,  $U$  is basepoint free, and  $I_U$  has a linear first syzygy, then the determinant of the degree  $\nu = (2a - 1, b - 1)$  submatrix of the first differential in the approximation complex is determined by  $\{L, S_1, S_2\}$ .*

*Proof.* This follows from Lemma 1.3, Lemma 1.4, the remarks preceding those lemmas, and Theorem 2.2  $\square$

**Corollary 2.4.** *If  $a, b \geq 2$ ,  $U$  is basepoint free, and  $I_U$  has a linear first syzygy, then the singular locus of  $X_U$  contains a line.*

*Proof.* Let  $I_U = \langle pu, pv, p_2, p_3 \rangle$ . By Corollary 2.3, the matrix representing the degree  $\nu$  component  $d^1$  has as its leftmost  $2a(b-1)$  columns a block matrix  $P$ . For each monomial  $m_c = s^{2a-1-c}t^c$  with  $c \in \{0, \dots, 2a-1\}$ , there is a  $b \times b-1$  block  $B$  corresponding to elements  $m_c \cdot \{v^{b-2}, \dots, u^{b-2}\} \cdot L$ , with  $L = vx_0 - ux_1$ , hence

$$B = \begin{bmatrix} x_0 & 0 & \dots & \dots & 0 \\ -x_1 & x_0 & 0 & \vdots & 0 \\ \vdots & -x_1 & \ddots & \vdots & 0 \\ \vdots & 0 & x_0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & -x_1 & x_0 \\ 0 & 0 & 0 & 0 & -x_1 \end{bmatrix}, \text{ and } P = \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & B \end{bmatrix}.$$

Computing the Laplace expansion of the determinant using the  $2ab - 2a$  minors of  $P$  shows the implicit equation for  $X_U$  takes the form

$$x_0^{2ab-2a} \cdot f_0 + x_0^{2ab-2a-1} x_1 \cdot f_1 + \dots + x_1^{2ab-2a} \cdot f_{2ab-2a}.$$

So  $X_U$  is singular along  $\mathbf{V}(x_0, x_1)$ , with multiplicity at least  $2ab - 2a$ .  $\square$

### 3. APPLICATION TO THE BIDEGREE $(2, 2)$ CASE

In this section we give some examples in the bidegree  $(2, 2)$  case; without loss of generality we assume  $I_U$  has a linear first syzygy of bidegree  $(0, 1)$ , so  $I_U = \langle pu, pv, p_2, p_3 \rangle$ . Hence  $p$  is of bidegree  $(2, 1)$ ; if  $p$  is not irreducible, there are two possible factorizations for  $p$ . If  $p$  is a product of linear forms, then identifying the coefficients of  $p = a_0 s^2 u + a_1 s t u + a_2 t^2 u + a_3 s^2 v + a_4 s t v + a_5 t^2 v$  with a point of  $\mathbb{P}^5$ , such a decomposition corresponds to a point on the Segre variety  $\Sigma_{2,1}$ , whose ideal is defined by the two by two minors of

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}.$$

The other possible decomposition occurs when  $p = ql$ , where  $q = a_0 s u + a_1 s v + a_2 t u + a_3 t v$  is irreducible of bidegree  $(1, 1)$  and  $l = b_0 s + b_1 t$  of bidegree  $(1, 0)$ . This is the image of the map

$$\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0))) = \mathbb{P}^3 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^5,$$

$(a_0 : a_1 : a_2 : a_3) \times (b_0 : b_1) \mapsto (a_0 b_0 : a_0 b_1 + a_2 b_0 : a_2 b_1 : a_1 b_0 : a_1 b_1 + a_3 b_0 : a_3 b_1)$ , which is a quartic hypersurface

$$Q = \mathbf{V}(x_2^2 x_3^2 - x_1 x_2 x_3 x_4 + x_0 x_2 x_4^2 + x_1^2 x_3 x_5 - 2x_0 x_2 x_3 x_5 - x_0 x_1 x_4 x_5 + x_0^2 x_5^2).$$

Note that  $\Sigma_{2,1} \subseteq \mathbf{V}(Q)$ . We now give three examples of possible bigraded betti tables for these three situations, where  $p_2$  and  $p_3$  are chosen generically. It would be interesting to prove that these are actually always the bigraded betti tables for generic choices of  $p_2$  and  $p_3$ , and to study what bigraded resolutions are possible in the  $(2, 2)$  case. We are at work on this project. For brevity, we denote  $R(a, b)$  by  $(a, b)$ . In all three cases,  $X_U$  has degree  $2ab = 8$ , in contrast to Example 1.1.

$$0 \leftarrow I_U \longleftarrow (-2, -2)^4 \longleftarrow \begin{array}{c} (-2, -3) \\ \oplus \\ (-4, -3)^2 \\ \oplus \\ (-4, -4) \\ \oplus \\ (-3, -5)^2 \\ \oplus \\ (-6, -3) \\ \oplus \\ (-8, -2) \end{array} \longleftarrow \begin{array}{c} (-4, -5)^3 \\ \oplus \\ (-6, -4)^2 \\ \oplus \\ (-8, -3)^2 \end{array} \longleftarrow \begin{array}{c} (-6, -5) \\ \oplus \\ (-8, -4) \end{array} \longleftarrow 0$$

**Example 3.2.** Now suppose  $p \in \mathbf{V}(Q) \setminus \Sigma_{2,1}$ . After a change of coordinates, we may assume  $p$  is the point  $(1 : 2 : 1 : 1 : 0)$ , which corresponds to  $s^2u + 2stu + t^2u + s^2v + stv$ .

**Example 3.3.** Now suppose  $p \in \Sigma_{2,1}$ . After a change of coordinates, we may assume  $p$  is the point  $(1 : 1 : 1 : 1 : 1 : 1)$ , which corresponds to  $s^2u + stu + t^2u + s^2v + stv + t^2v$ .

The reduced singular locus of  $X_U$  consists of curves of degrees 1 and 4.

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<http://www.math.uiuc.edu/Macaulay2/>

and scripts to perform the computations are available at

<http://www.math.uiuc.edu/~schenck/Syzscript>

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